INTRODUCTION TO INVERSION IN MUSIC

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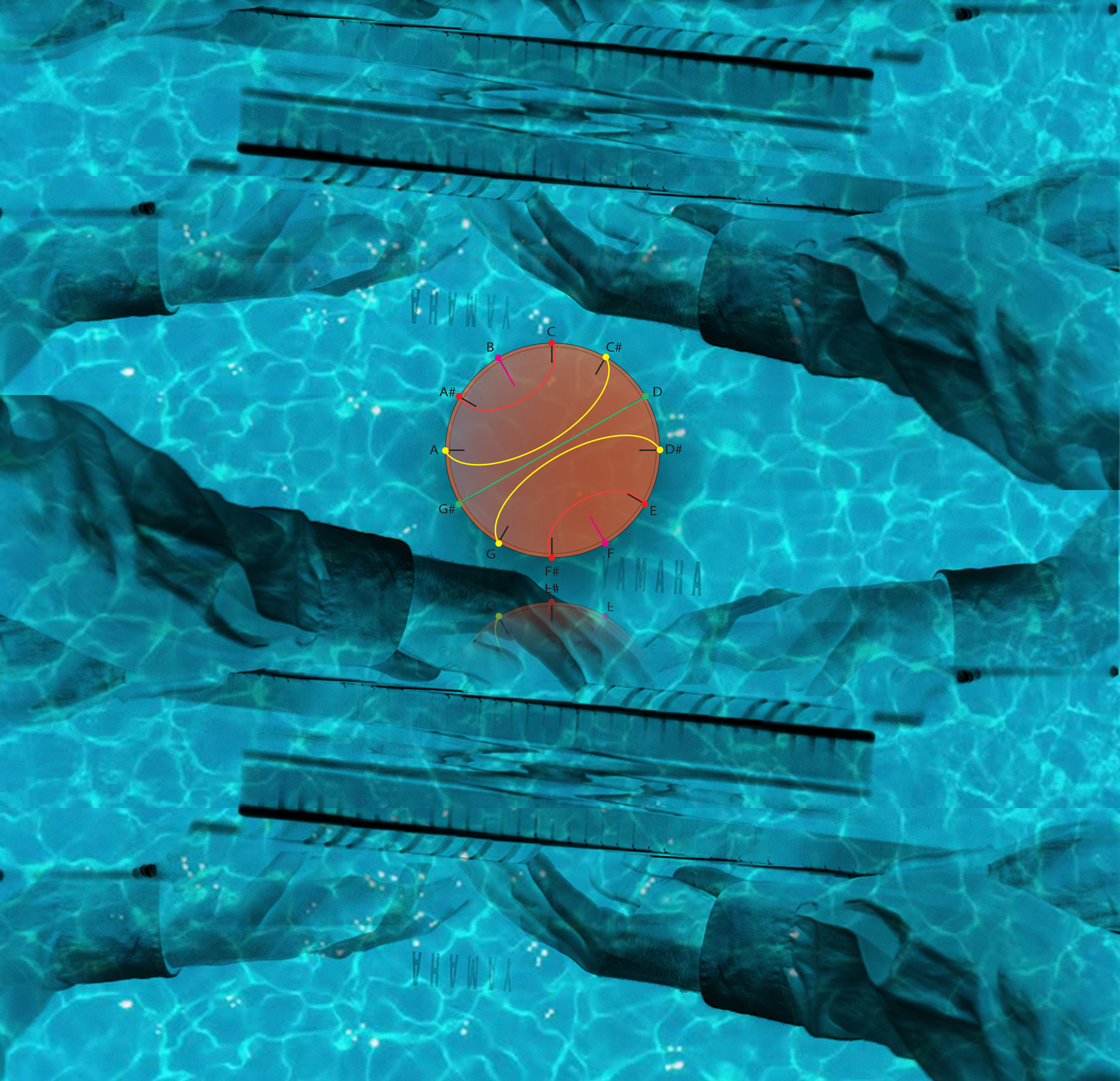

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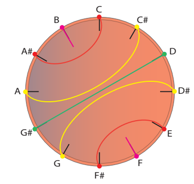

INTRODUCTION

Inversion is a concept that is found broadly within the musical fields of music theory, counterpoint, harmony, composition, aural training, notation, and the practice of musical instruments. It is based upon the over-arching concept of symmetry – a trait found in all the arts.[1]

This paper aims to alert the musician of the multitude of interpretations of the term *inversion*, to amalgamate different approaches and terminology into one reference document, as well as to inspire the reader to further their understanding via the research and publications found in cited authors' works.

Inversion can be thought of as a principle of variation. Its application may create related variants in rhythm, melody and harmony that create difference whilst promoting unity.[2] The superposition of inversionally-related material with its corresponding original material may create rich and interesting tapestries of rhythm, harmonic complexity and melodic counterpoint.

Whilst even the trained musician may at times experience difficulty in accurately or aurally perceiving the procedures at work in the creation of inverted rhythmic, melodic or harmonic material, inversion nevertheless contributes to at least an intuitive sense of congruence in music.

[1] See Dimond, 2019 for an introduction to the concept of symmetry and approaches to its musical application.

[2] Schillinger considers inversion of pitch space simply one of the four *geometrical inversions* that arise from rotations of melody around axes of pitch and time, which he illustrates using graphs (185). These correspond to the basic procedures of original, retrograde, inversion, and retrograde-inversion found in post-tonal music.

TERMINOLOGY AND ENQUIRY

Inversion can broadly mean to turn upside-down, to reverse, or to turn something into its opposite. In applying the term to music, it is useful to be reminded of the existence of a range of musical parameters to which the term may apply. These include pitch – both as a continuum pitches with differentiation across registers, and as a periodic octave-equivalent sequence of pitch-classes – rhythm, form/structure, dynamics, and-so-on.

During any one day of a music student's training, they may encounter different and possibly contradictory uses of the term *inversion* as they move from composition seminar to aural musicianship class, from music theory and harmony class to their private instrumental lesson. Recognising the context in which the term is used within these different situations is necessary so that the intended application may be understood. Furthermore, an awareness of implicit assumptions made in each context may lead the musician to their conception of new possibilities for the application of the principle of inversion, along with the development of heretofore unknown techniques.

From its symmetric roots, inversion can be represented easily in two-dimensional space using a diagram such as that in Figure 1. As with any symmetric operation, the *axis of reflection* needs to be acknowledged. In the case of this diagram, the inversion occurs about the x-axis (horizontal), indicated by a dotted pink line. It could be said that the symbol (L) is shown alongside its inverted form, and also *vice versa*.

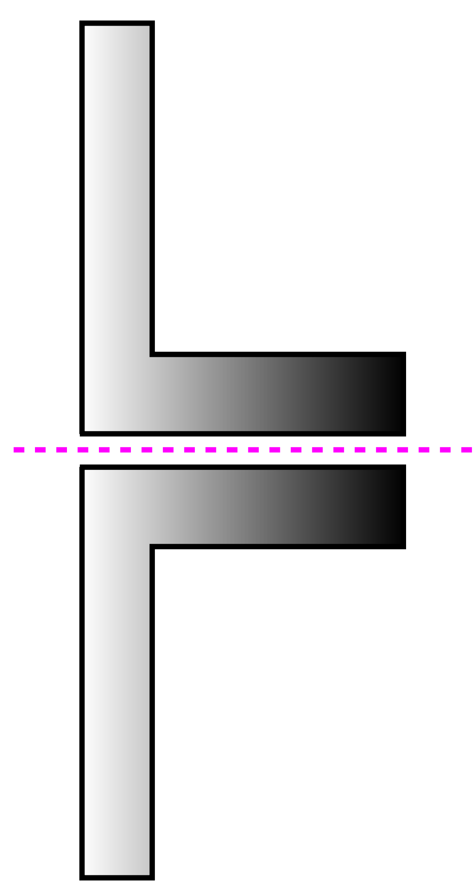

Figure 1. Inversion as x-axis (horizontal) bilateral symmetry.

Figure 1 illustrates the kind of inversion most familiarly applied in melody and harmony. What exactly are the axes representing? What is the musical result of the axis of reflection, and the space around it? The distance between two reflected forms of inversionally symmetric objects is called the *interval of reflection*.

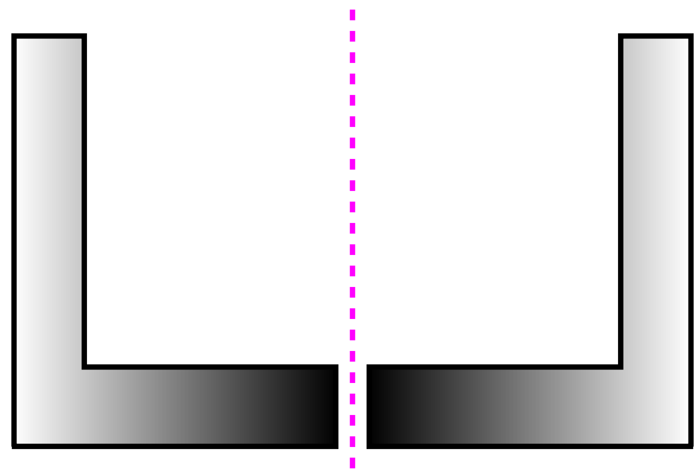

Figure 2. Inversion as y-axis (vertical) bilateral symmetry.

Figure 2 illustrates another kind of inversion often applied to melody and rhythm, though not to harmony. What again are the assumptions we are making about the parametric application of the axes, and again what is the musical result of the axis of reflection?

Whilst we are in a mode of enquiry, what might the following inversionally-symmetric sequence represent?

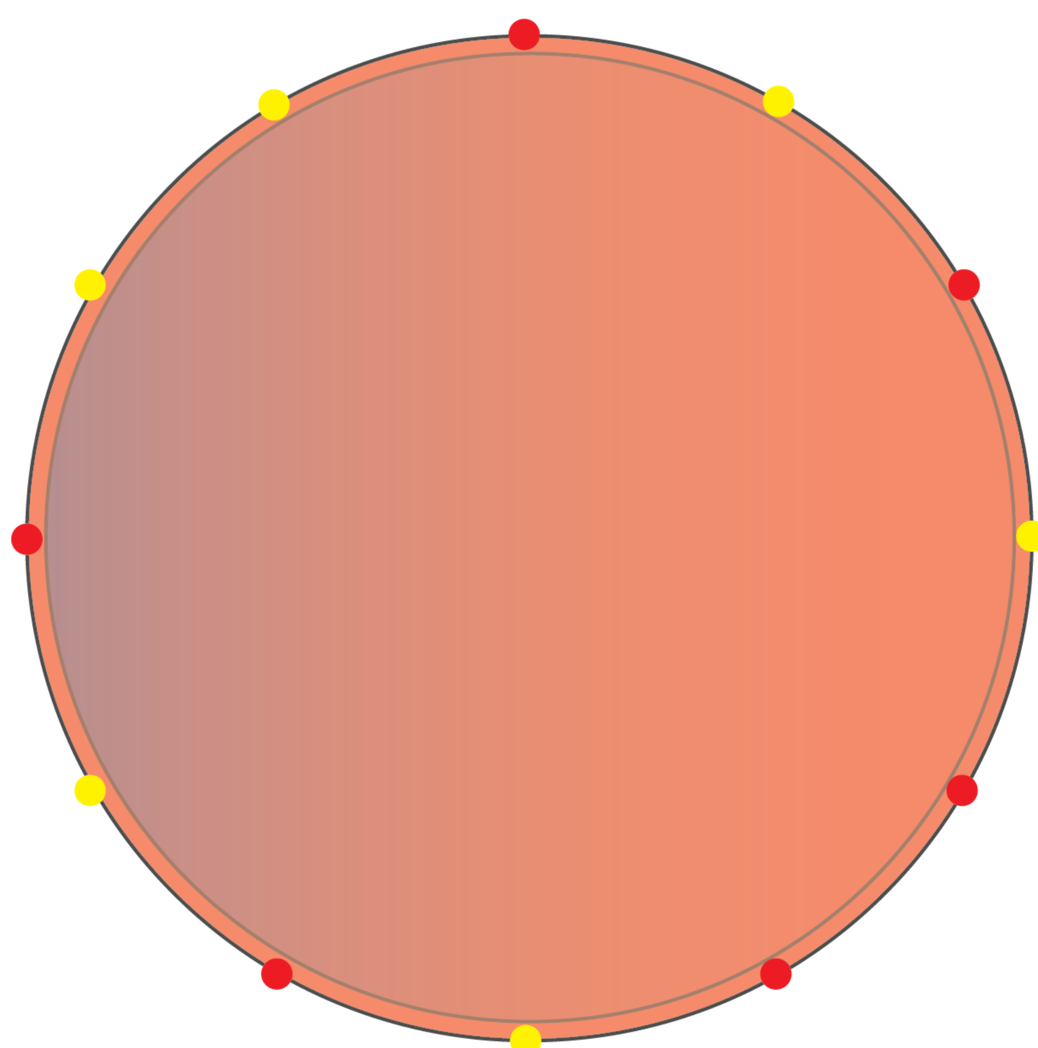

Figure 3. Bracelet pattern containing inversional symmetry.

Figure 3 illustrates a kind of inversional symmetry found in combinatorial mathematics, whereby two groups or sequences are considered equivalent if they are the reverse of each other. That is, you can remove a bracelet from your hand and replace it backwards, resulting in an inversion of the patterning, as distinct from the rotation of a necklace around your neck. Binary patterns such as in the bracelet illustrated are typically interpreted as rhythmic onsets in music,[3] though they may also be interpreted in the pitch domain as representing complementary pitch-class sets.[4]

[3] See Toussaint for more on necklaces and bracelets (Toussaint 73) as well as the creation of reflection rhythms and rhythmic canons from such symmetries (207).

[4] See Rahn (140) for a table of pitch class sets and their complements as well as a discussion of inversionally-symmetrical pitch class sets (90). Okazaki (5) diagrammatically represents the same table in his 'Table of Melodic Elements'.

INVERSION IN PITCH SPACE

Consider pitch as a continuum that traverses the audible frequency range, as illustrated in the piano keyboard of Figure 4.[5]

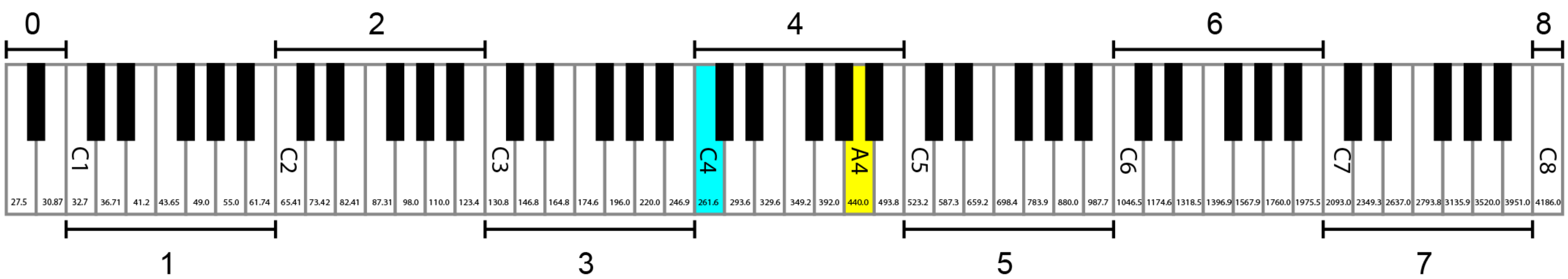

Figure 4. Piano keyboard with octave registers and frequencies.

If a specific pitch is designated as an axis of reflection, all pitches may be said to posses inversional relationships with pitches equidistant from that pitch on the opposite side of the axis. Figure 5 nominates Bb4 as the axis of reflection. Some of the resulting inversional relationships in pitch space are notated in Figure 6.[6]

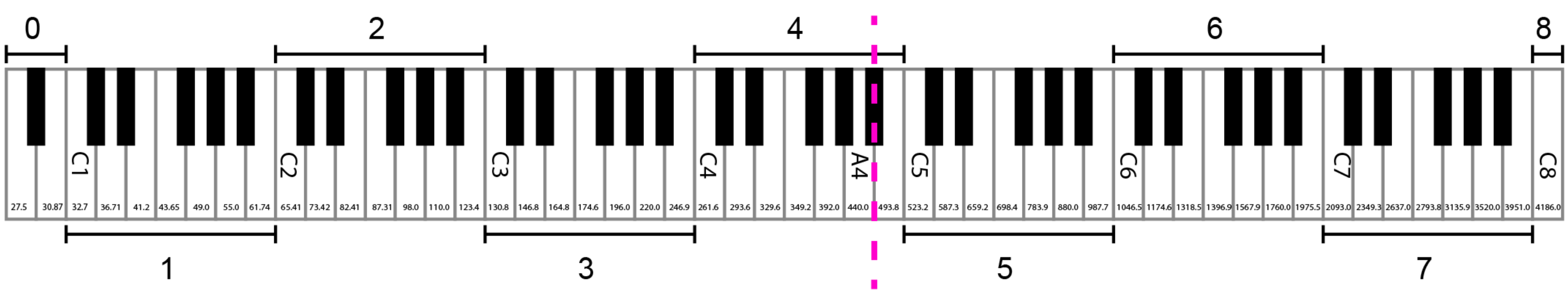

Figure 5. Piano keyboard with axis of reflection about Bb4.

This kind of pitch centricity is common in post-tonal music, and precisely this axis is chosen by Bartók in the closing moments of the first movement of his *Fifth String Quartet* (Straus 232). An *inversional wedge* is a melodic application of such inversional symmetry in pitch space, whereby melodic lines may converge upon or expand from the axis of reflection to create cadential or distending effects in the texture. These can be categorised as *contracting* or *expanding* inversional wedges, respectively (Straus 234).[7]

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[5] Middle C and the standard orchestral tuning note are colour-coded blue and yellow respectively, for orientation.

[6] Bilateral symmetry creates unisons at the octave either side of the axis of reflection every half-octave, or tritone.

[7] Lendvai discusses the "funnel-shaped" expansion or contraction of intervals in Bartók's chromatic language (75). See also Rahn (50).

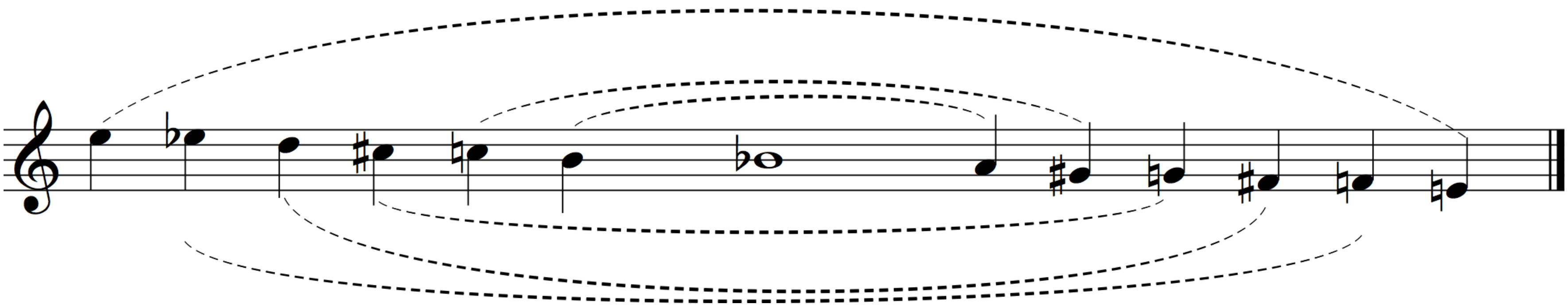

Figure 6. Notated inversion in pitch space with axis of reflection about Bb4.

It is worth indicating that axes of reflection need not place centricity upon a specific pitch in a register, but may alternatively centralise a dyad straddling either side of the axis by lying upon a pitch midway between two semitones. Figure 1 diagrammatically represented an inversion possessing a certain interval of reflection. Compare this with Figure 7, where no such interval exists. Figure 7 graphically represents the notation of Figure 6, where an inversional wedge may converge upon a specific pitch, such as Bb4.

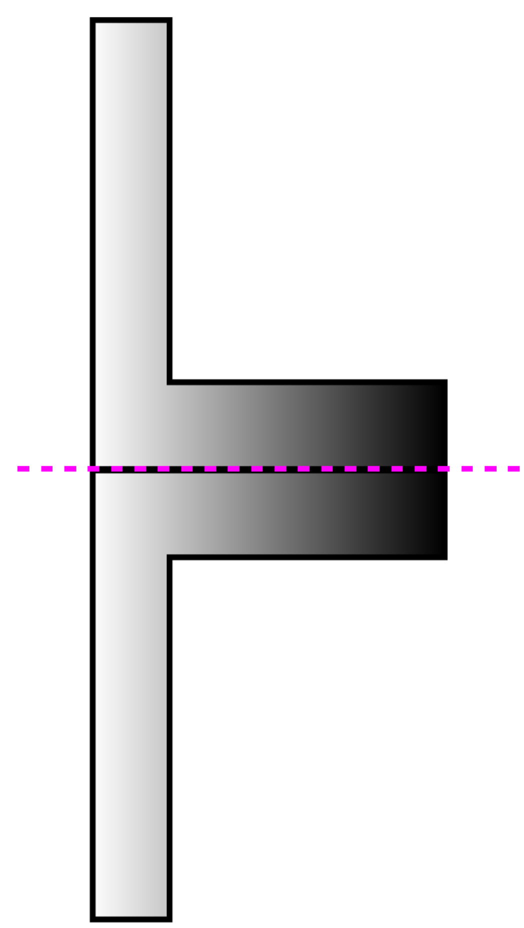

Figure 7. Inversion as x-axis (horizontal) bilateral symmetry, with zero interval of reflection.

Figure 8 applies the circumstance where a certain interval of reflection exists, per Figure 1. In this harmonic example, Schoenberg superimposes two augmented chords, whose axis of reflection is effectively a quarter of a semitone sharp of E5. This chord occurs in the end of movement 2 from *Six Little Piano Pieces*, op.19.

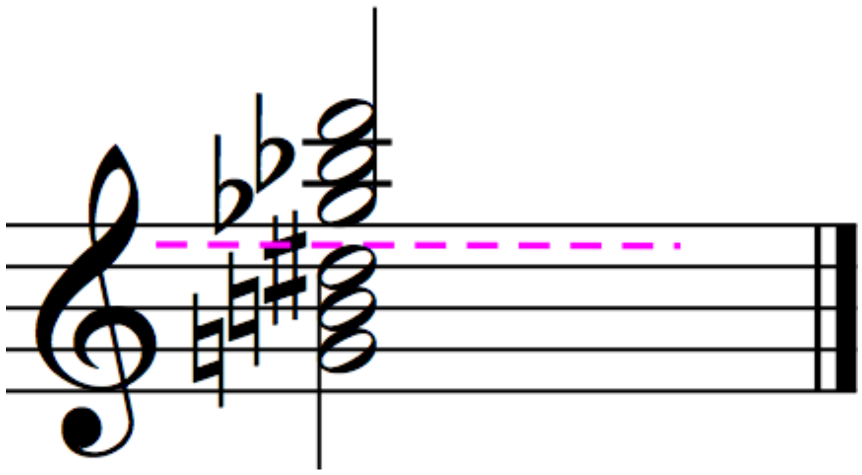

Figure 8. Inversion as x-axis (horizontal) bilateral symmetry about the dyad E5-F5, with interval of reflection of 1 semitone around E5 quarter-sharp.

The tool for adjusting the interval of reflection is transposition, because transposition effectively maps interval content by a set distance.[8] Take for example the melody in Figure 9, which assigns a pitch axis to C4.

Figure 9. Inversion as x-axis (horizontal) bilateral symmetry about C4, with zero interval of reflection.

In Figure 9 the pitches of each part reflect the opposite part by the same interval above/below C4. Figure 10 illustrates some additional options where an inversional wedge effect is less prominent due to the increasing interval of reflection. The bass clef part is progressively undergoing transposition by semitones downward.[9]

[8] Both transposition and inversion preserve relative distances of the intervals, when applied strictly. They correspond to the symmetric operations of *translation* and *reflection,* respectively. Refer to Dimond 2019 for a summary of symmetric operations and their musical application using geometry as inspiration.

[9] Inversion is thought of as a composite operation of reflection and transposition. The bass clef part in Figure 10 is said to be inverted and then transposed, with transpositions at T-1, T-2 and T-3 respectively.

Figure 10. Inversion as x-axis (horizontal) bilateral symmetry, with interval of reflection incrementing by semitone about the dyads B3-C4, Bb3-C4, and then A3-C4.

In this way, the choice of interval of reflection effects the harmony implied, as illustrated by the bitonal effect of the middle system compared to the diatonic effect (lacking any need for accidentals) of the bottom system of Figure 10.

So far we have considered pitch space as a chromatic continuum. Inversional effects may still be created whilst diatonic adjustments to pitch space are made to serve harmonic considerations. Figure 11 takes the treble clef idea of Figure 9 and forces the result into C major. In this manner, an inversional wedge can still be achieved whilst serving specific harmonic requirements.

Figure 11. Inversion as x-axis (horizontal) bilateral symmetry, with zero interval of reflection about C4 and diatonic adjustment.

Inversional mappings such as that of Figure 11 can be ascertained by notating the desired scale or mode instead of the chromatic collection (of Figure 6). Just as before, transposition can be applied in such a way that also complies with the harmonic requirements of the music. Figure 12 represents the inversional relationships of Figure 11, whilst Figure 13 provides an example of a diatonic mapping with an interval of reflection (dyadic axis of reflection).

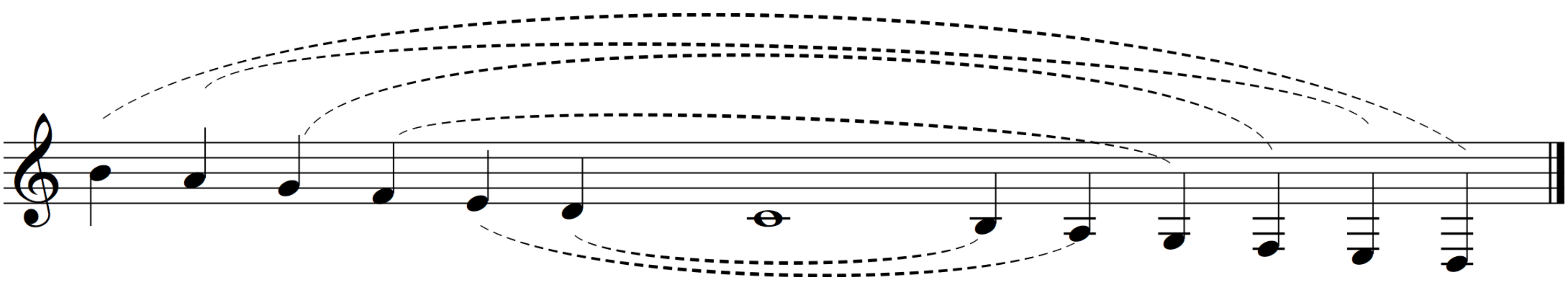

Figure 12. Notated inversion in pitch space with axis of reflection about C4 and diatonic adjustment.

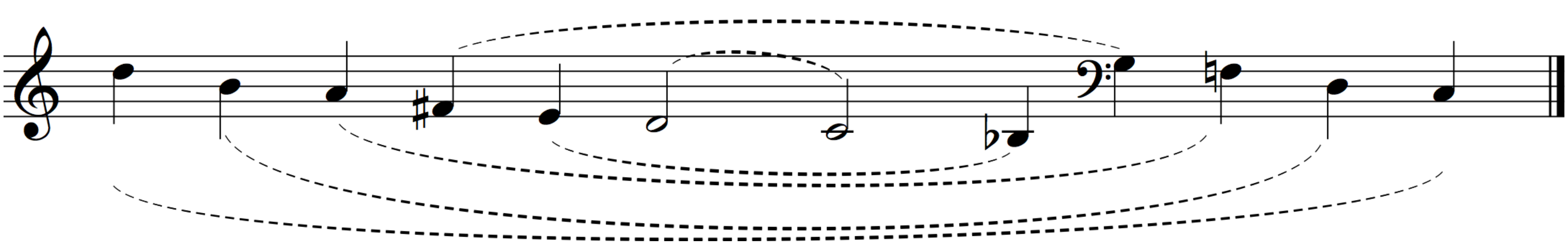

Figure 13. Notated inversion in pitch space with axis of reflection about dyad C4-D4 and diatonic adjustment.

The pitch material in Figure 13 is pentatonic, but orientates two modes around the two transpositions, D major pentatonic (mode 1) in the top with Bb major pentatonic (mode 2, from C) in the bottom. Endless possibilities exist in ways to form such inversionally-related modes and scales in pitch space, that can be further refined by choices of registration and orchestration.

These examples demonstrate that inversion in chromatic pitch-space need not manifest in post-tonal 12-tone music. The composer has only to apply their imagination to create the relationships they seek. Take the modes of the diatonic scale for example. Figure 14 illustrates that the intervals of the ionian mode correspond to the phrygian mode when the intervallic series is inverted.[10]

Figure 15 identifies the inversional relationships all of the modes of the diatonic scale accordingly, illustrated using notation and their retrograde interval sequences. Note that the

[10] Melodic and harmonic minor are alternative scale sources. For more on this approach refer to Schillinger's 'geometrical inversions' (198).

combined pitch collection of both heptatonic modes does not necessarily represent all 12 chromatic pitches. Ionian and phrygian omit the tritone (Gb in the given key). Dorian, with a non-retrogradable interval sequence, maps to itself and so does not introduce any new pitches.[11]

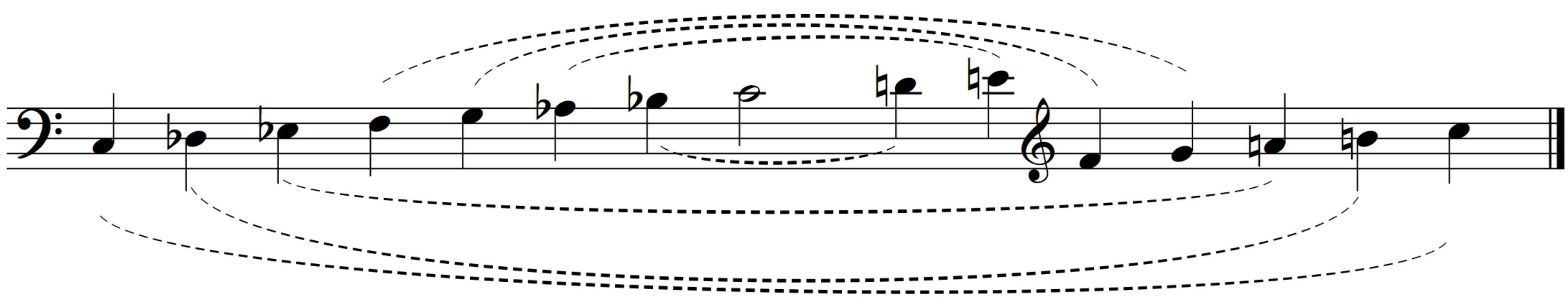

Figure 14. Ionian and phrygian as inversionally-related modes in pitch space with axis of reflection about C4.

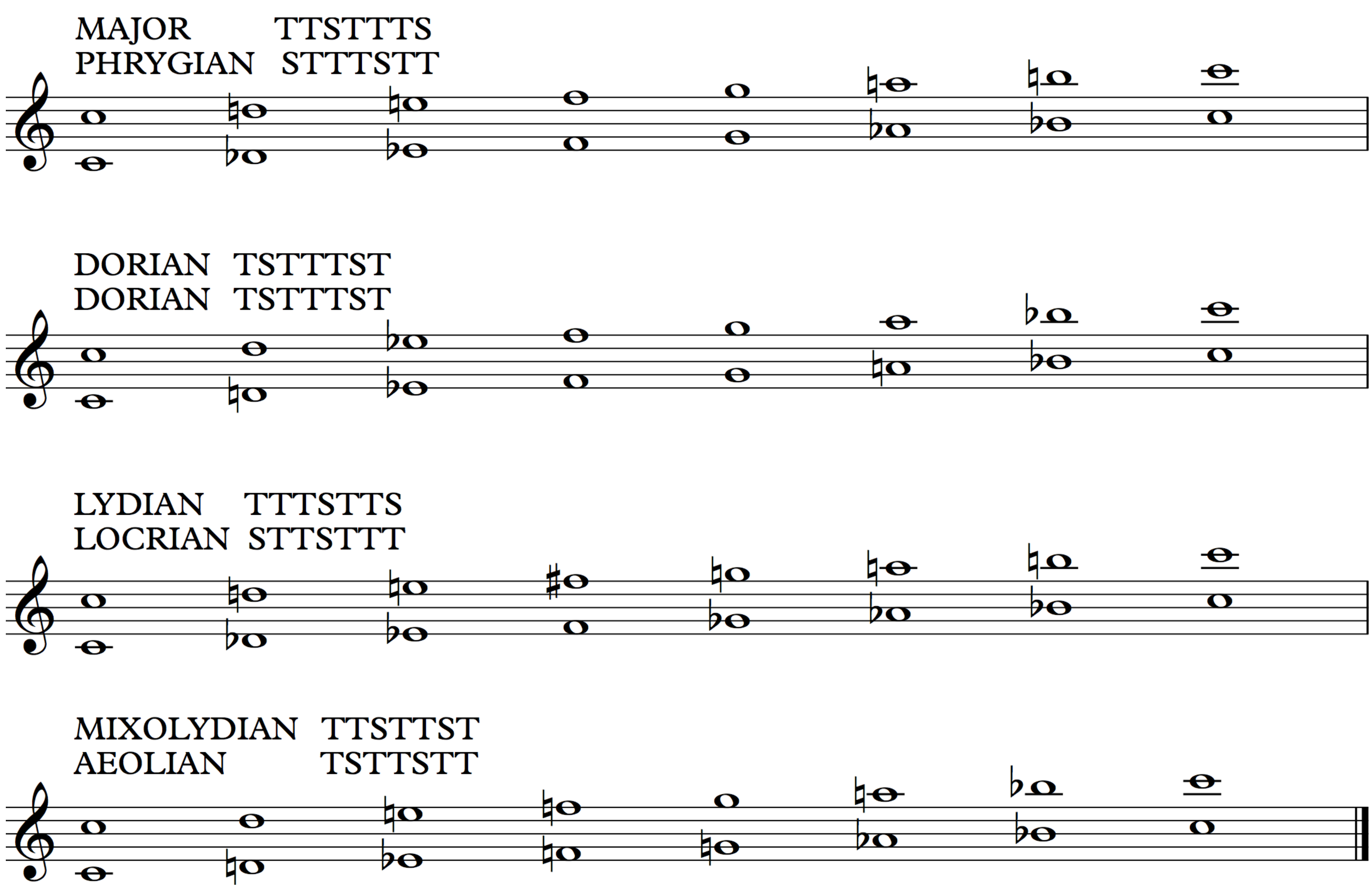

Figure 15. All modes of the major scale with inversionally-related modes notated ascending in parallel, with their interval series (T=tone, S=semitone).

[11] Experimentation with intervals of reflection would create all sorts of key relationships and different pitch collections.

INVERSION IN PITCH REGISTER

A contradictory use of the term *inversion* can be found in aural training, harmony and some music theory classes, which refers to a change in octave of one or more pitches in a chord.[12] Such cases of *registral inversion* occur during the consideration of close or spread chord voicings, complementary intervals, cyclic permutations of pitch order, and during the analysis and realisation of *figured bass* notation in compositions of the Baroque era. Such applications of the term are categorically different to the strict distance-preserving symmetries of reflection in pitch space about an axis of reflection (whether exact or diatonically tempered), and also to reflections in pitch-class space (discussed in the next section).

Figure 14 illustrates the three common *positions* of the major triad – root, first and second, respectively. If we try and see determine how reflection in pitch space is operating in such inversions, we could only deduce that an axis of reflection a tritone above the bottom note operates in a discriminate manner in each triad, applying to only the bottom note to produce the subsequent inversion. In short, no such distance-preserving symmetry exists, so this is not *inversion* as we have defined it previously.

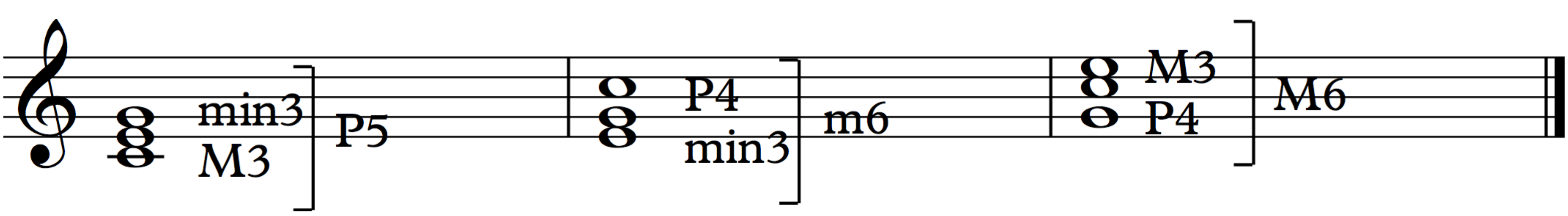

Figure 14. Major triad with pitch register inversion and resultant intervals.

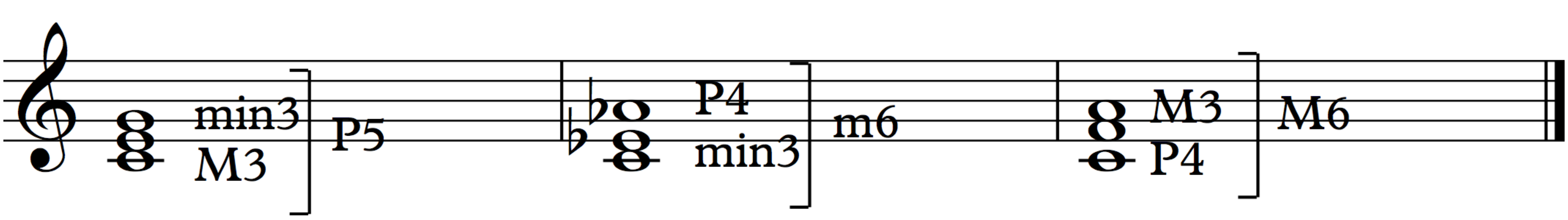

Figure 15. Major triad with pitch register inversion and resultant intervals, all transposed to C4.

Figure 15 illustrates the same triad and its transformations, but also uses transposition to maintain a common bass note for the root, first and second inversion triads, respectively.

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12 Inversion of pitch register through conventional cyclic permutation can only occur for structures smaller than an octave, such as triads and 4-note chords.

What this approach emphasises is the lack of intervallic congruence during registral inversion. Each triad can only be mapped to each other by octave change of one pitch and the transposition of the three.

The difference between pitch register and pitch space inversion is further illustrated in Figure 16, where the same C major root position triad is inverted about C4 using the aforementioned procedure of pitch space inversion. Interestingly, the resulting triad is a minor quality, and highlights the similarities in intervallic structure for two chords often considered intrinsically opposite – major and minor qualities might represent contrasting tonalities but it is the order or hierarchy, rather than the content, that forms the basis of such a contrast.

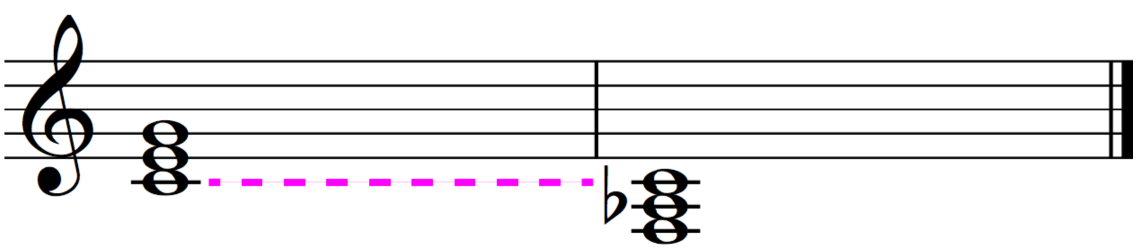

Figure 16. Major triad and minor triad related by pitch space inversion about C4.

INVERSION IN PITCH-CLASS SPACE

Pitch-class space takes a one octave segment from the continuum of pitch space and wraps it into a circle. Figure 17 illustrates such a periodic, chromatic circle of pitch classes.[13] The result is that all pitches of a certain letter name (e.g. C#) are categorically equivalent, regardless of actual octave placement in the music.[14] Movement from one pitch class to another can be enacted by either clockwise or anticlockwise motion around the circle, and every pitch-class interval has a nearby and more distant measurement, depending on whether an ascending or descending route is taken.[15] In this way, pitch-class space is more abstract than pitch space, which means it can be a powerful tool for composition and analysis, as deep structures in harmony and melody can inform great variety of realisation.[16]

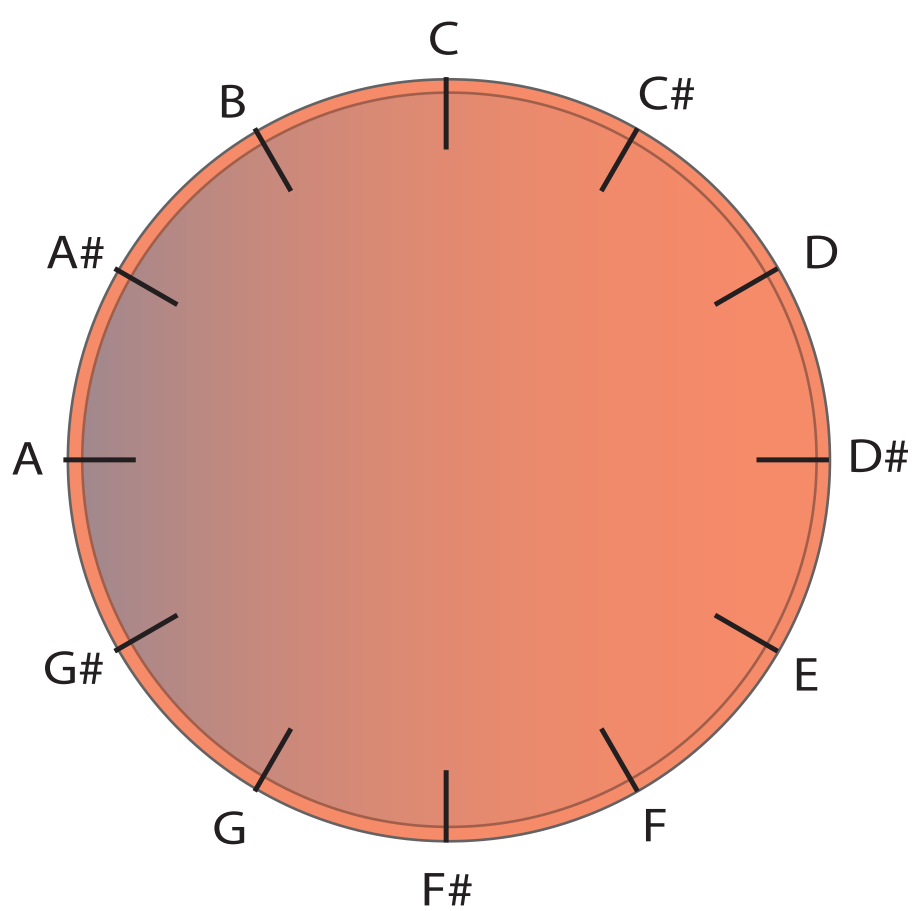

Figure 17. Circular pitch-class space.

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[13] In post-tonal music theory pitch classes are typically represented by integers from 0 (representing C) through 11 (representing B). See Rahn for more about the integer model of pitch (19).

[14] Octave equivalence is considered almost universal across global musical traditions (Tymoczko 30).

[15] Except for the tritone, being the half-octave distance between two diametrically opposed pitch classes. There are categories for the different kinds of pitch-class interval, being ordered (ascending) or unordered (the shorter of the descending or ascending). See Rahn (25).

[16] This paper limits our discussion to that of inversion, but for a broader study of pitch class refer to Rahn, Schillinger, Straus and Tymoczko.

Pitch-class inversion concerns bilateral symmetry in pitch-class space. This requires an axis of symmetry with end points a tritone (half-octave) apart. There are twelve such diametrically-opposed end points on the pitch-class circle. These are illustrated in Figure 18.

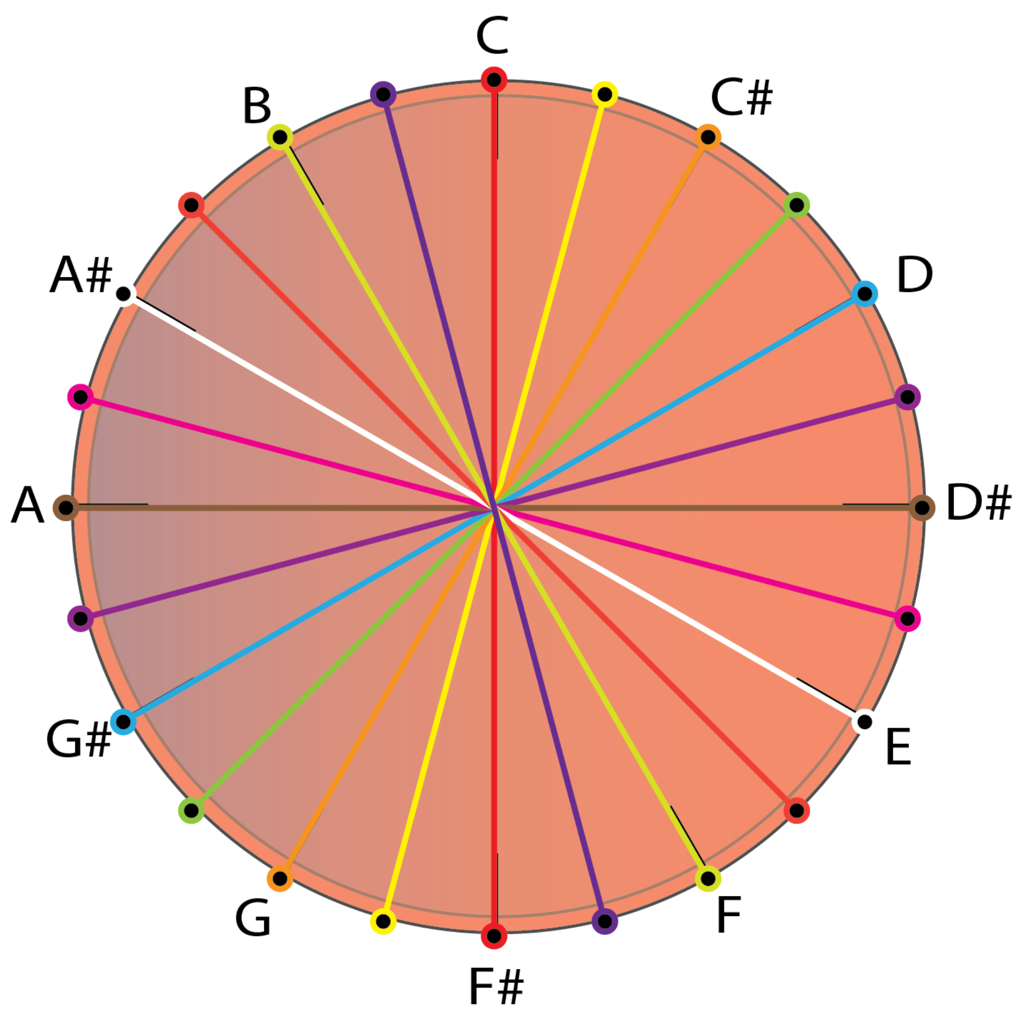


Figure 18. Twelve possible axes of inversion in pitch-class space.

Figure 18 reveals the possibility that the axis of symmetry formed by diametrically-opposed tritones may fall between two chromatic pitches. In such cases of pitch-class inversional symmetry, the two pitch classes that straddle the end-points of the axes possess an inversional relationship, whilst in cases where the axes falls on a specific pitch class, these map to themselves.[17] Figure 19 illustrates the two scenarios respectively, taking the yellow and blue axes from figure 18. The dotted lines show the inversional mappings in pitch-class space.

[17] In either case, these tritone-related end-points are considered the two centres of the inversional symmetry (Rahn 50).

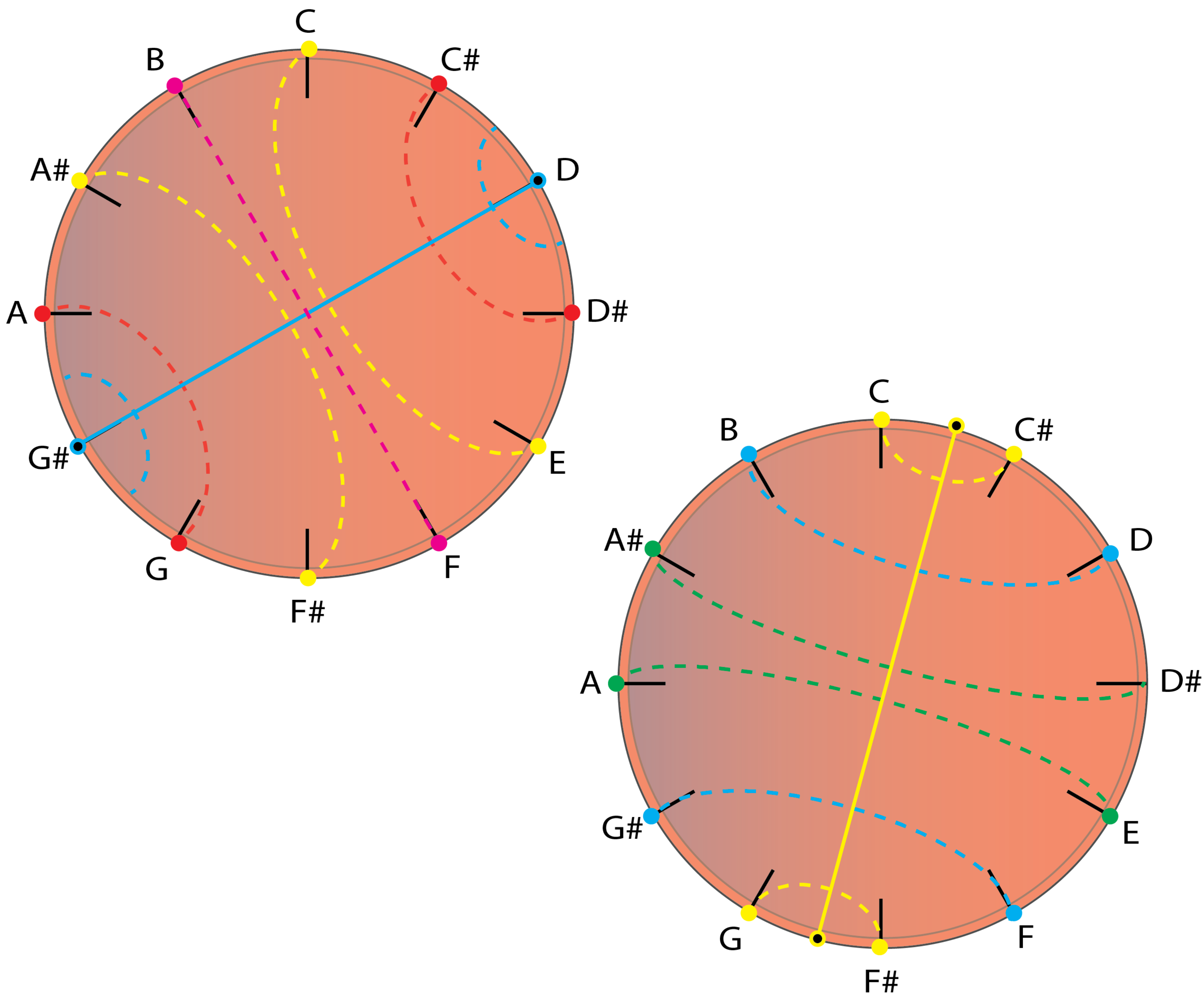


Figure 19. Two of the twelve possible axes of inversion in pitch-class space showing inversional mappings about these axes (dotted lines).

Whilst Figure 19 may illuminate the possibilities for inversionally-symmetric melodic motion in 12-tone pitch-class space, this kind of diagram may also be used to create harmonic structures that are inversionally symmetrical. Figure 20 plots a pentachord (0,1,3,4,8) whose intervals display bilateral symmetry about the blue axis of Figures 18 and 19. When rotated, the ordered pitch-class intervals of this set are invariant under retrograde operation – a hallmark of inversional symmetry.

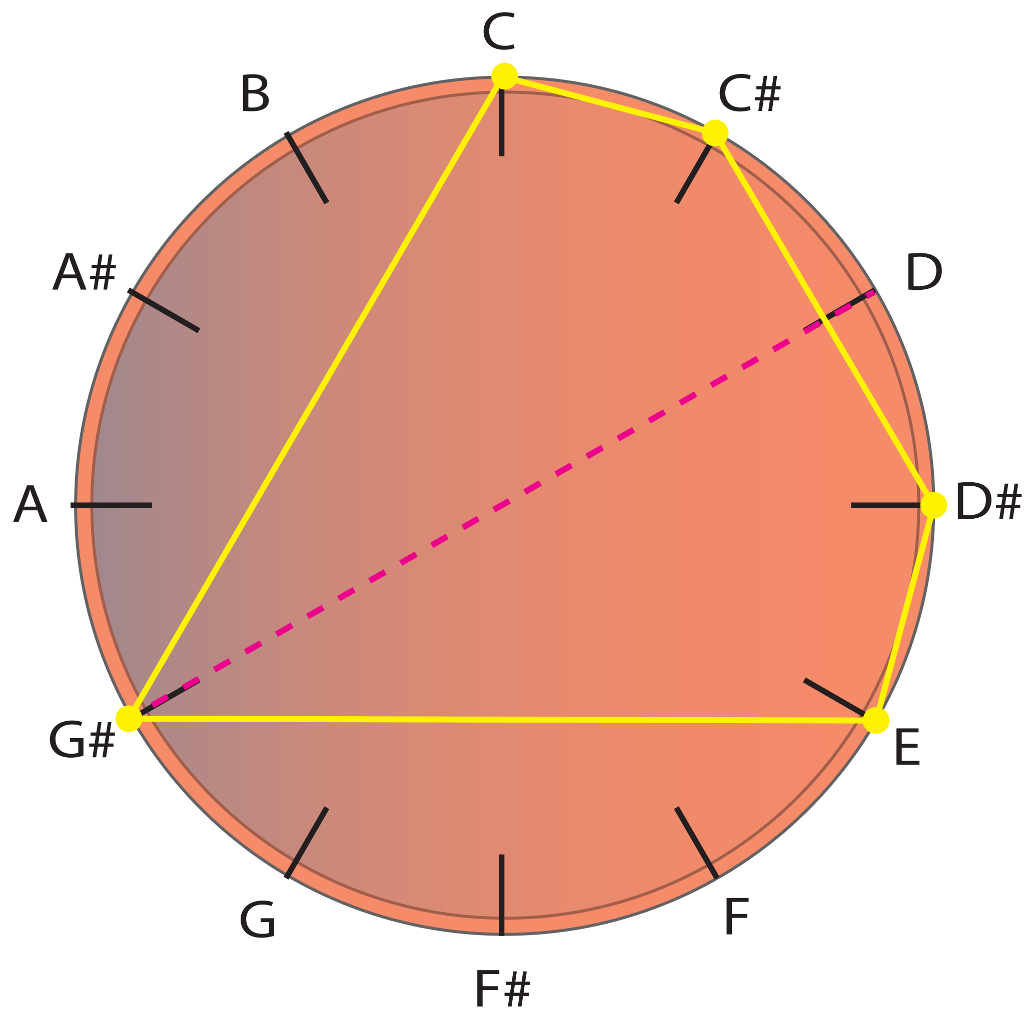


Figure 20. The inversionally-symmetric pentachord (0,1,3,4,8) with ordered pitch-class intervals 4-1-2-1-4.

Compositions possessing inversional symmetry about an axis of symmetry held by a tritonic division of pitch-class space may evade analysis using the previous pitch space approach, as variations in octave placement may obscure the presence of inversional symmetry. Straus cites various examples from Bartok and others, showing that compositional meaning is educed more readily through this approach in circumstances where octave changes and changes of pitch centricity about the opposite poles of the axis may occur over time (Straus 236).[18]

[18] Straus also shows that motion from one of the twelve axes to another may form a logical type of "modulation" within such compositions (241).

COMPARISON OF INVERSION IN PITCH SPACE, REGISTER AND PITCH-CLASS SPACE

A better understanding of inversion in pitch-class space may be obtained through comparison with the previous categories of inversion.

The diminished triad is commonly referred to as a symmetric structure. However, from the perspective of registral inversion, being the approach common in arranging and aural training contexts, this is not the case. This is because the interval content changes between the three positions of the diminished triad as one traverses registers. Each inversion possesses unique interval series. Distance is not preserved, which is a prerequisite for the strict definition of inversional symmetry. The first and second inversion diminished triads fail to orientate around an axis of reflection. Figure 21 plots out registral inversion of the triads on a keyboard, representing pitch space.

Figure 21. Diminished triad in root position (from C), first (from Eb) and second (from Gb) inversions, illustrated in pitch space with attempts at delineating axes of symmetry (pink dotted lines from left to right indicate the mid-point of the three respective triads).

Figure 21 supports the finding that a triad (or any structure) with true registral inversional symmetry would require an equal division of the octave.[19]

In Figure 22 isosceles triangles represent the diminished triad in pitch-class space where a clear axis of symmetry can be found, being the diametrically-opposed tritone of A-Eb, C-F# and A-Eb, for the root, first and second inversion triads, respectively. The result is that the

[19] Consider the factors of 12 and the resultant inversionally symmetrical tritone, augmented triad, diminished seventh chord, whole-tone and chromatic scales.

three “inversions” of the diminished triad can be said to be categorically the same triad, being transpositions (geometric rotations) of each other.[20]

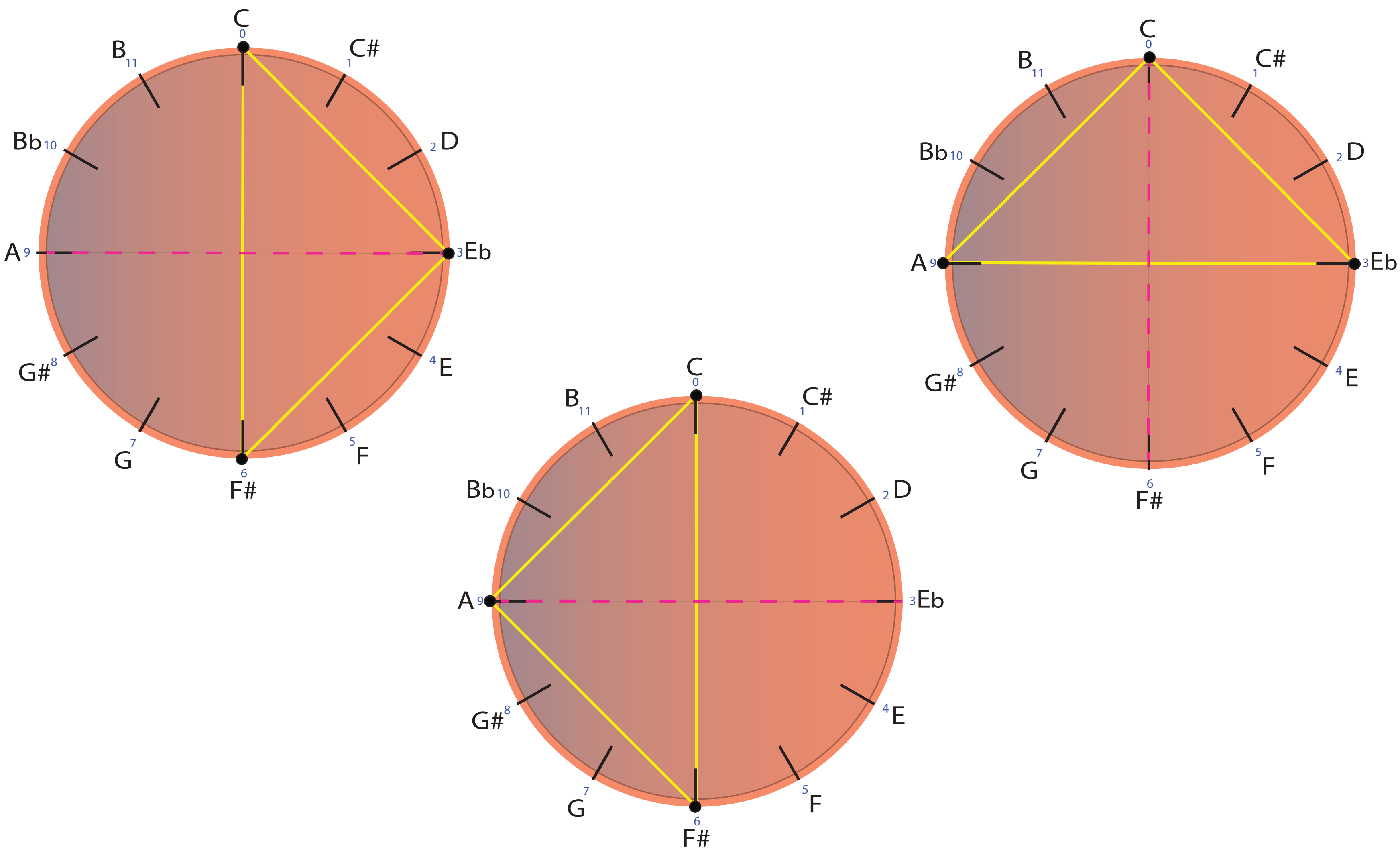


Figure 22. Diminished triad in root position (top left), first (top right) and second inversions, illustrated in pitch-class space with axes of symmetry (pink dotted line).

So whilst the diminished triad’s formations are invariant in pitch-class space, only root position is symmetric in pitch space, and registral inversion does not maintain the symmetry of the root position triad.

The same cannot be said for a triad asymmetric in pitch-class space, such as the major triad, whose ordered pitch-class intervals are 4-3-5, as first inversion does not equate to root position major in any key. There is no diametrically-opposed axis of symmetry able to be drawn through a scalene triangle. None of the three vertices correspond with a pitch midway between the other two when mapped in pitch space. Refer to Figures 23 & 14.

[20] Inversions of the C diminished triad effectively transpose the same series of ordered pitch-class intervals, being 3-3-6 (semitones) to different keys. The three diagrams of Figure 21 are considered *necklaces* of the same intervallic structure (in combinatoric mathematics). The result is that C diminished first inversion is the same as A diminished root position and C diminished second inversion is the same as F# diminished root position.

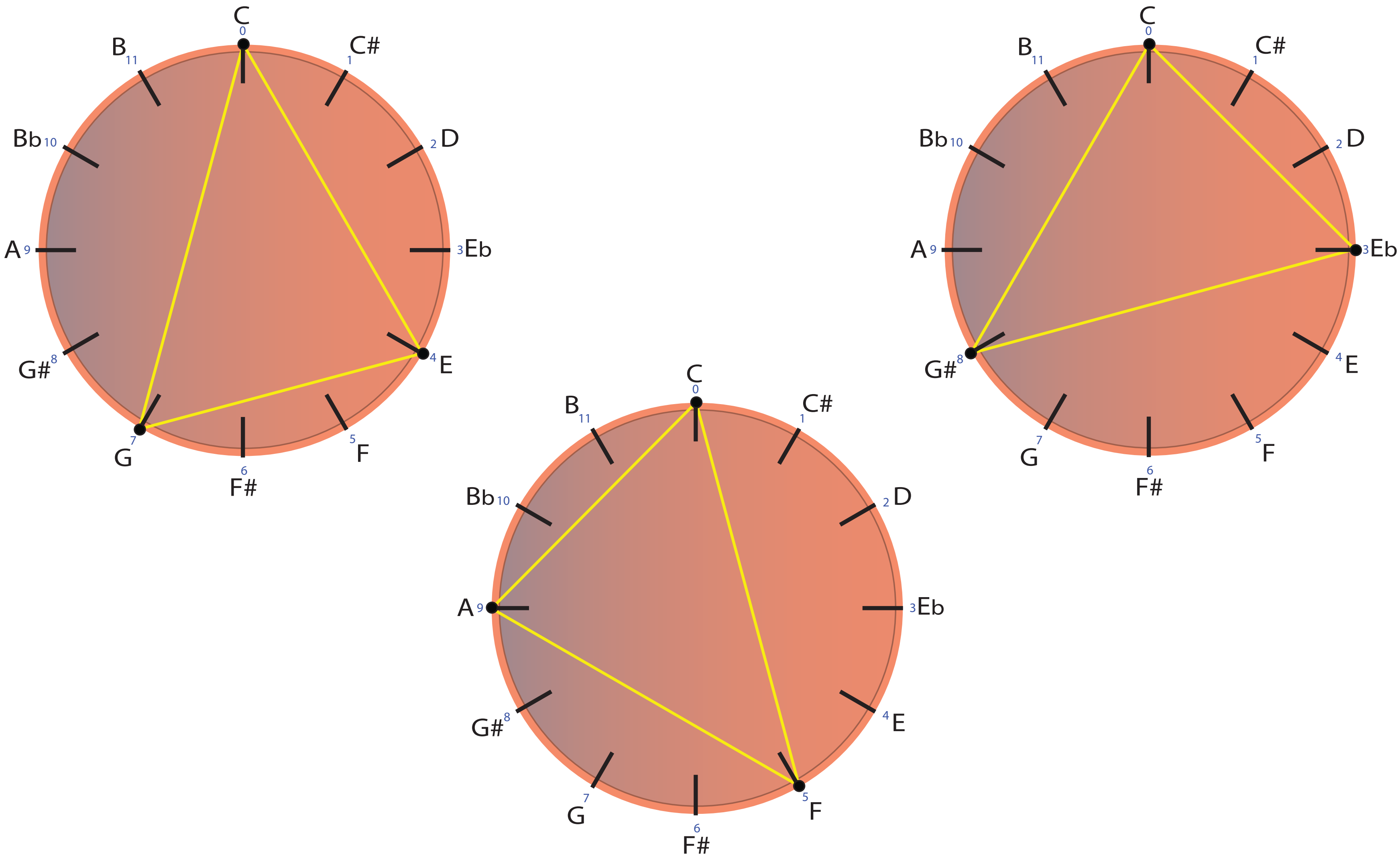
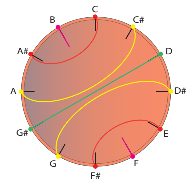


Figure 23. Major triad in root position (top left), first (top right) and second inversions, illustrated in pitch-class space.



INVERSION IN RHYTHM

Figure 3 presented a pattern consisting of two colours each with 6 appearances on a twelve-metric clock face. Figure 24 elucidates the inversional symmetry inherent in the pattern by outlining the sequence in the red and yellow polygons and showing the axis of symmetry.

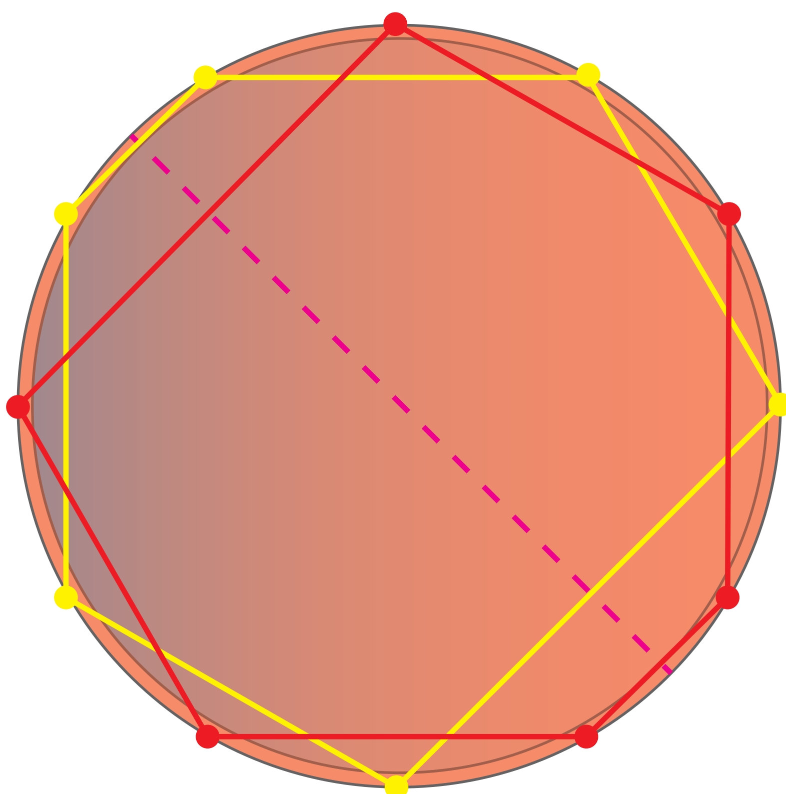


Figure 24. Complementary interlocking pattern from Figure 3.

Designating each intersection of the polygons with the circumference as a rhythmic onset, this pattern could be construed as a being in 12/8 metre, orchestrated in such a way that not only does a continuous pulse train arise from each layer being the opposite's complement, but the inversionally symmetric relationship of the two layers means that each layer is a displaced "echo" of the other.[21]

Toussaint describes such patterns as resembling *rhythmic canons,* whose effectiveness is optimised when the two layers possess contrasting timbre (215). Figure 25 notates the pattern of Figure 24.

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[21] Toussaint categorises these as *interlocking reflection rhythms,* and offers approaches to their design (207). The two polygons for this category of rhythm are said to be *bracelets* of each other, being inversionally related. (The same relationship can be acquired by *necklace* rotation of 50% of the timeline size.)

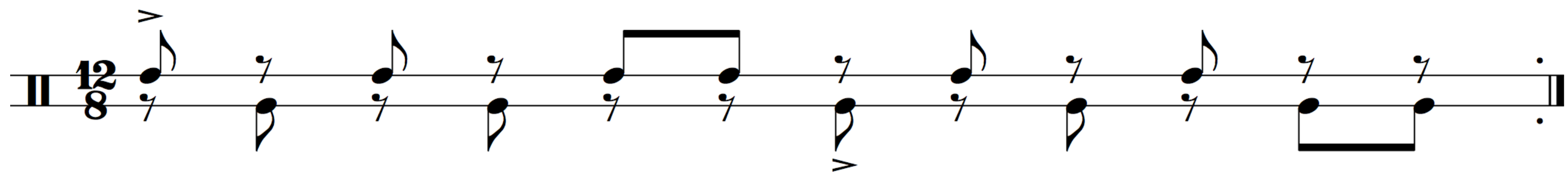

Figure 25. Interlocking reflection rhythm from Figure 24.

Toussaint offers approaches to the creation of such rhythms which include the *paradiddle* method (208). Single, double and triple paradiddles are example of common drum rudiments that possess the interlocking reflection rhythm property. Indeed by designating the red and yellow polygons of Figure 24 as right and left hands respectively, the double paradiddle pattern RLRLRRLRLRLL arises.

Whilst considering the options of inversionally-related rhythms on a 12-pulse timeline, one can turn to the theory of pitch-class sets to find the 20 possibilities of hexachords[22] whose complements map to the same set type. These were first tabulated by Allen Forte, and are reproduce by Rahn (142).[23] They are illustrated geometrically in Figure 26.

The nature of the axes of reflection is of interest (indicated by the pink dotted lines in Figure 26). As the mirror point about which the pattern is reflected, imagine folding each diagram along the pink dotted line to create a corresponding match with the opposite colour's polygon. Some patterns are inversionally symmetrical about more than one axis, such as sets (013679) or (014589). Also note the tendency for the axes to be orientated around 12 o'clock. This is because in set theory representative *prime forms* are always expressed "most packed to the left", the result of which is that the smaller inter-onset intervals of the original hexachord (represented by the red polygon) occur first, as one moves clockwise from zero. Logically, the inversion of this prime form hexachord (represented by the yellow polygon) will be most packed to the right.

Performing these inversionally-related rhythms on opposite hands repeatedly brings out the interlocking reflection nature of their patterning. Indeed, one develops a sense that the two parts are responding to each other, and that their inherent phraseology and interlock is more important than reference to a certain downbeat or metric hierarchy.[24] This revelation gives one a glimpse at the circular nature of the *timeline* in African music (Anku 1).

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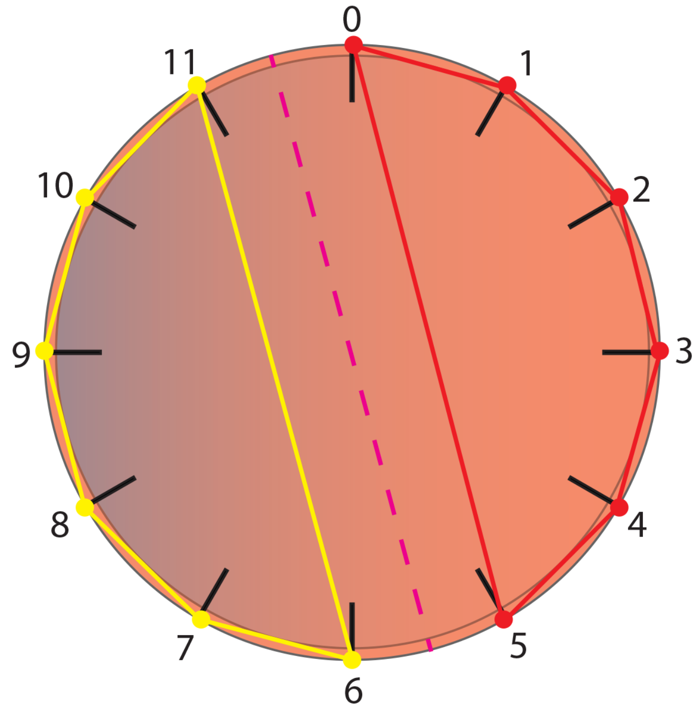
[22] It is only possible to create two-part rhythmic canons on timelines with an even number of pulses, due to the need for bilateral symmetry. Each layer requires exactly half the number of onsets.

[23] If a hexachord of a 12-tone row creates a 12-tone aggregate through the operation of inversion, retrograde, or retrograde inversion in some transposition, *hexachordal combinatoriality* is said to be present. The term is attributed to Milton Babbitt (Rahn 53).

[24] The reader is encouraged to perform the rhythms and all their associated *necklaces*.

(012345)



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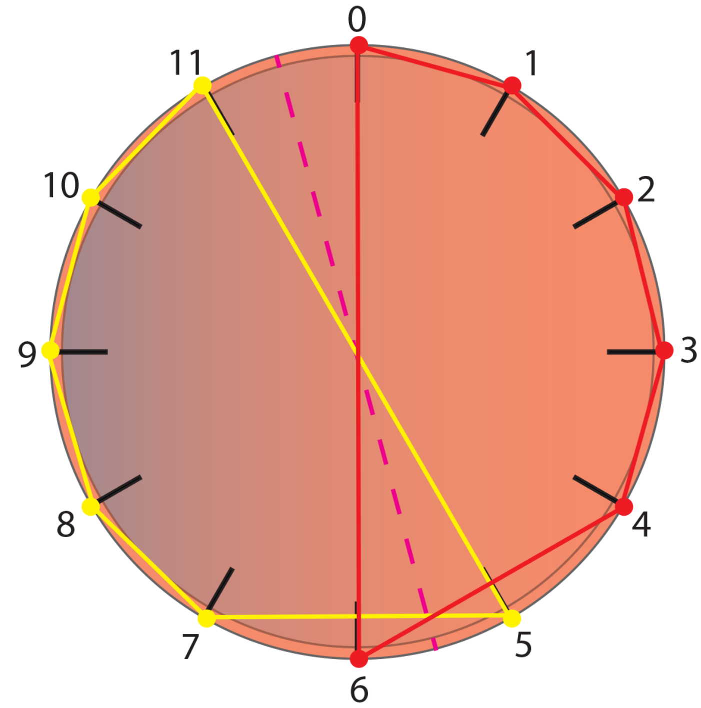

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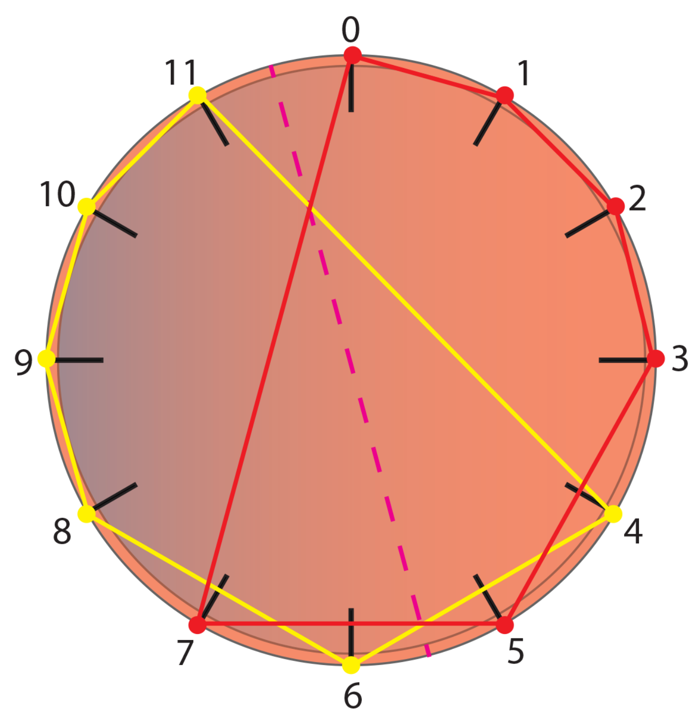

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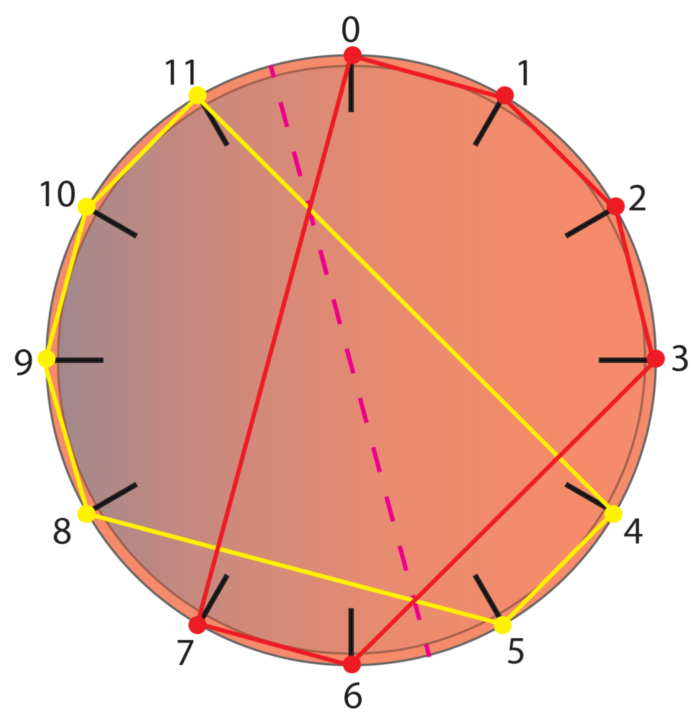

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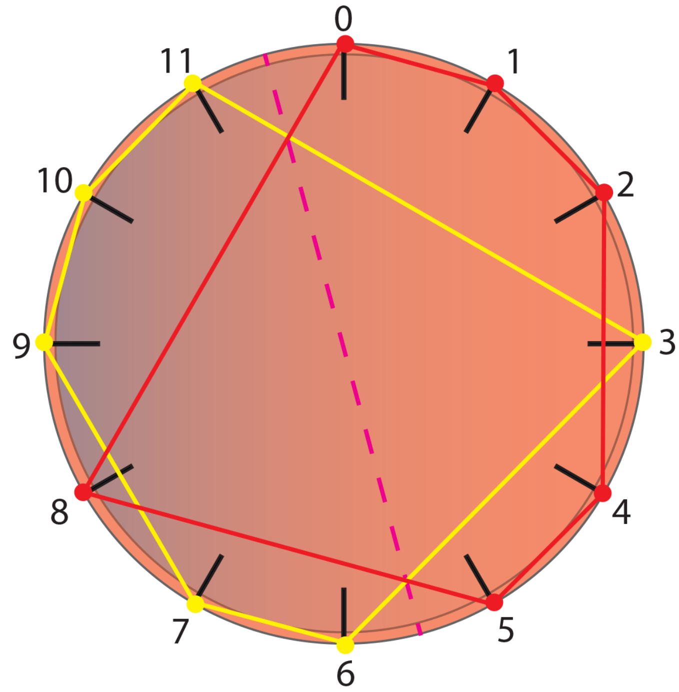

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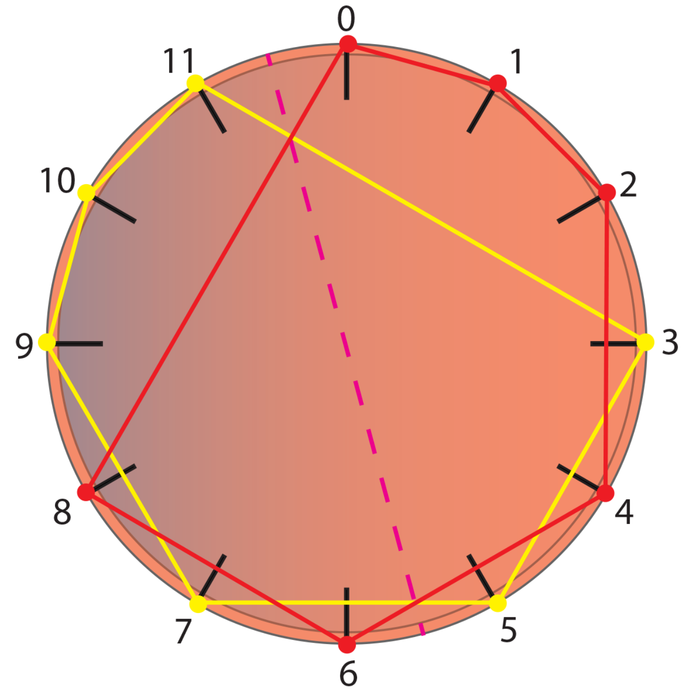

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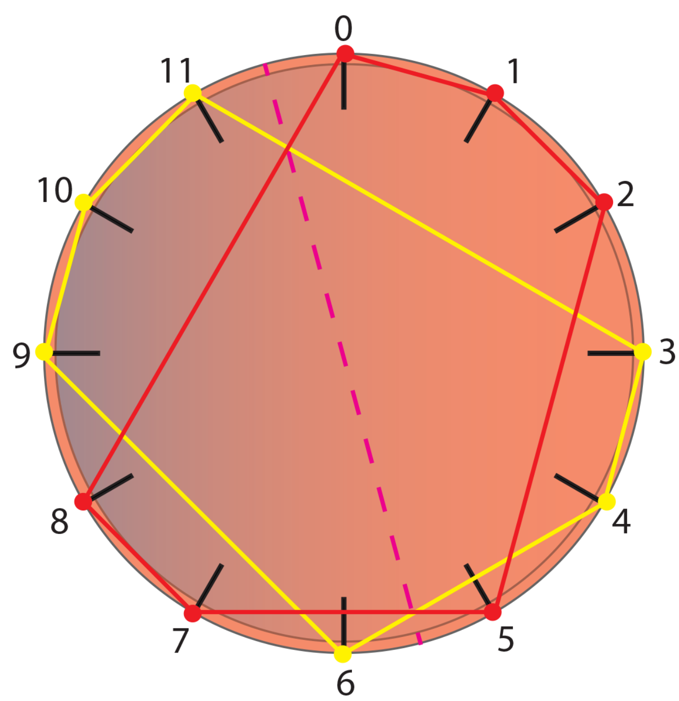

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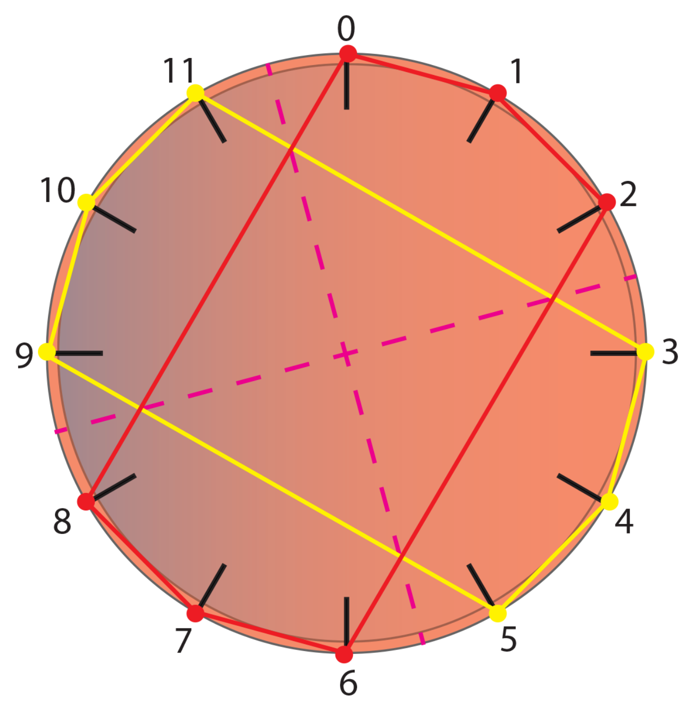

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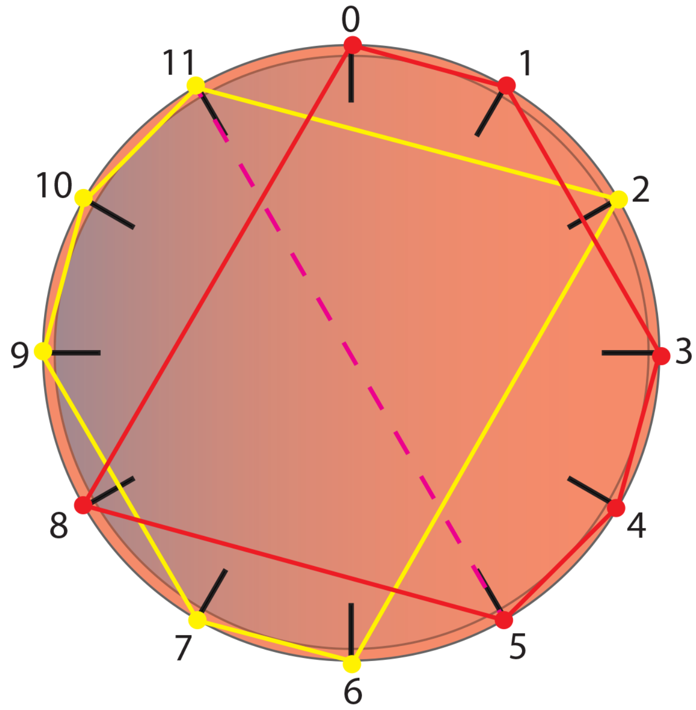

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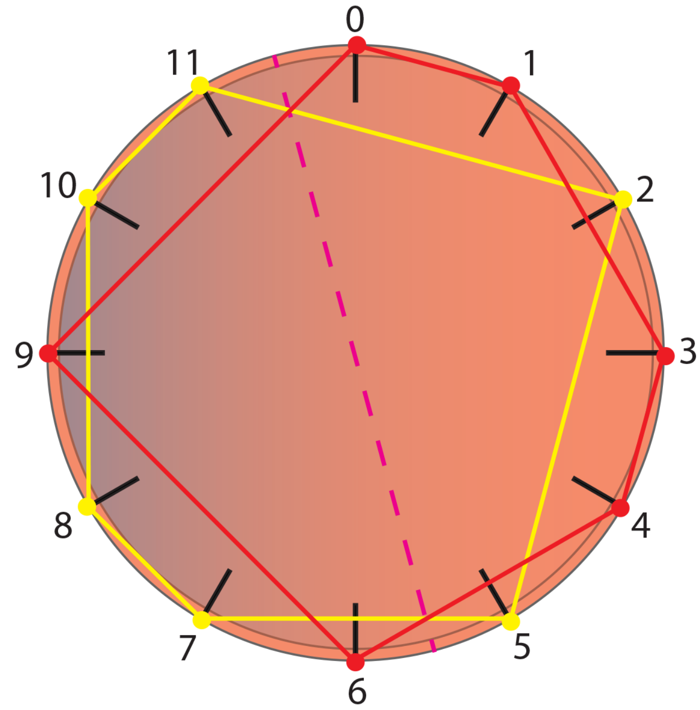

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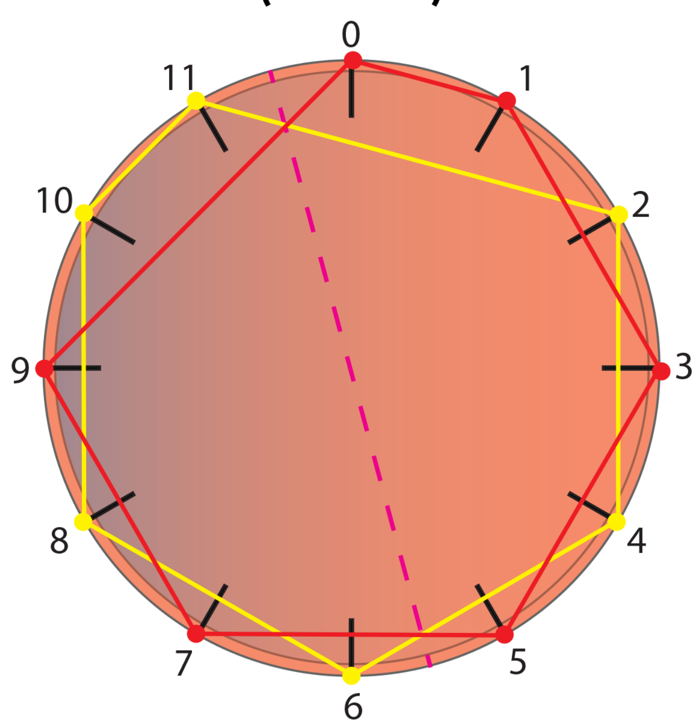

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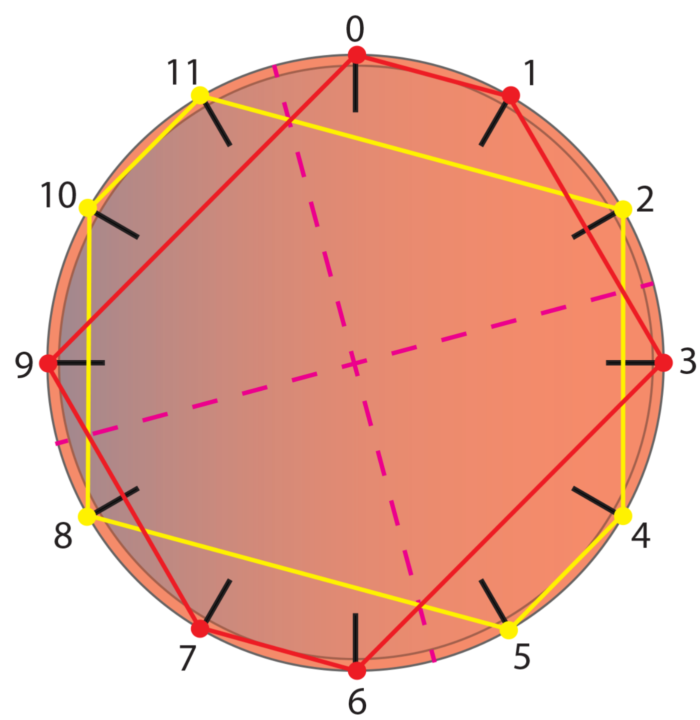

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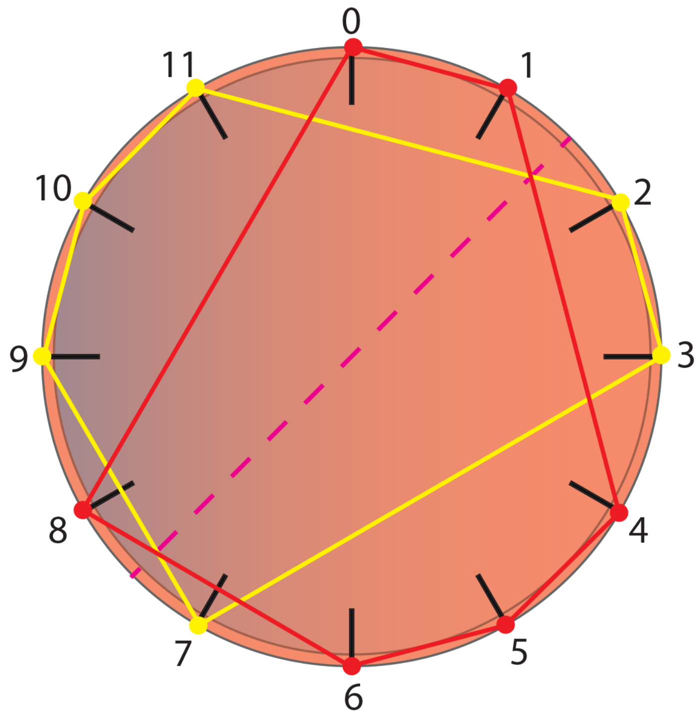

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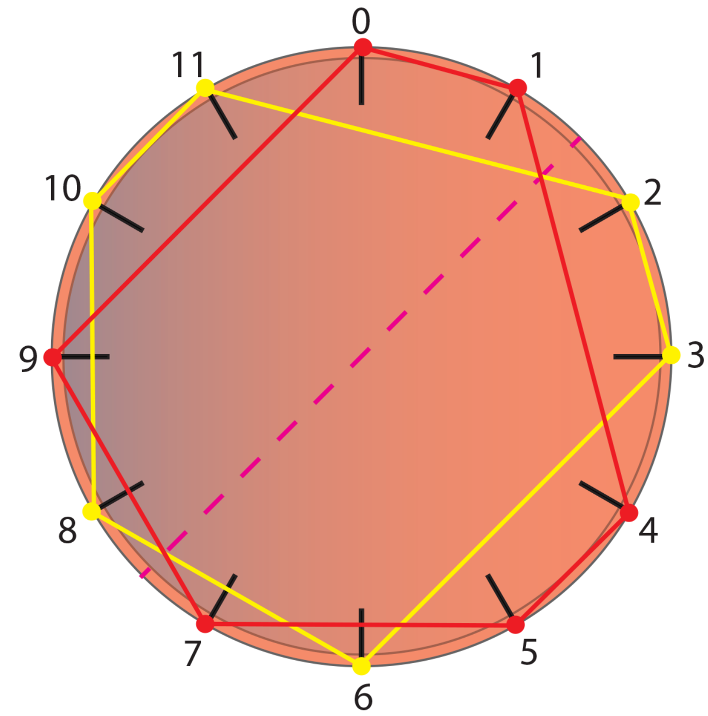

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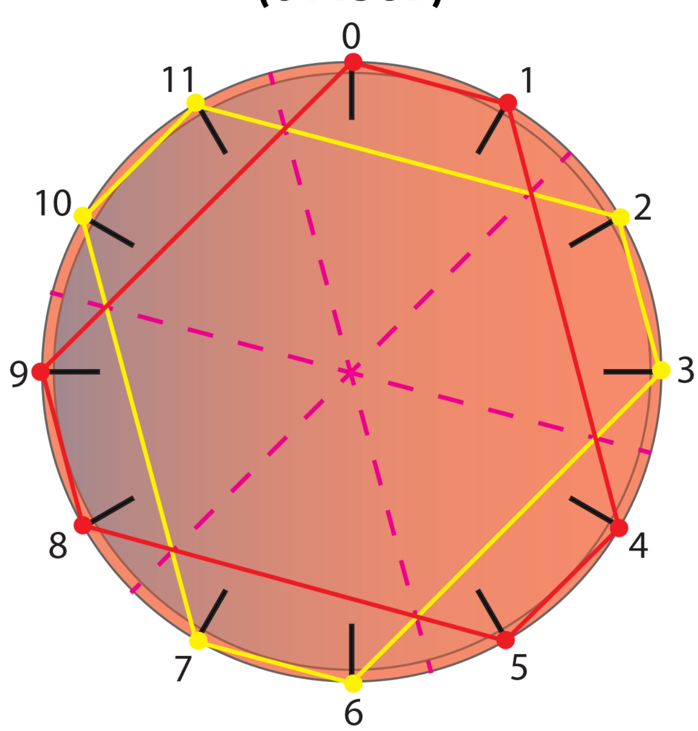

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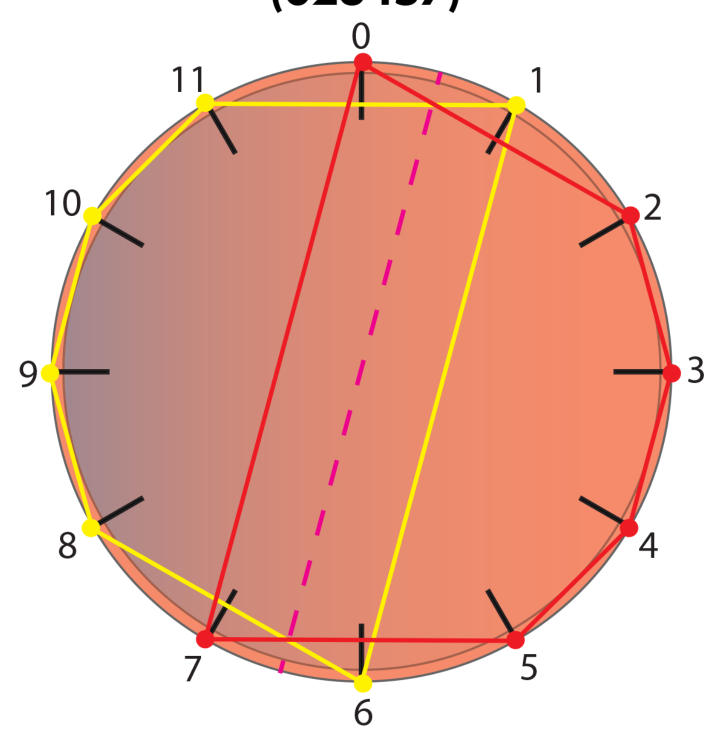


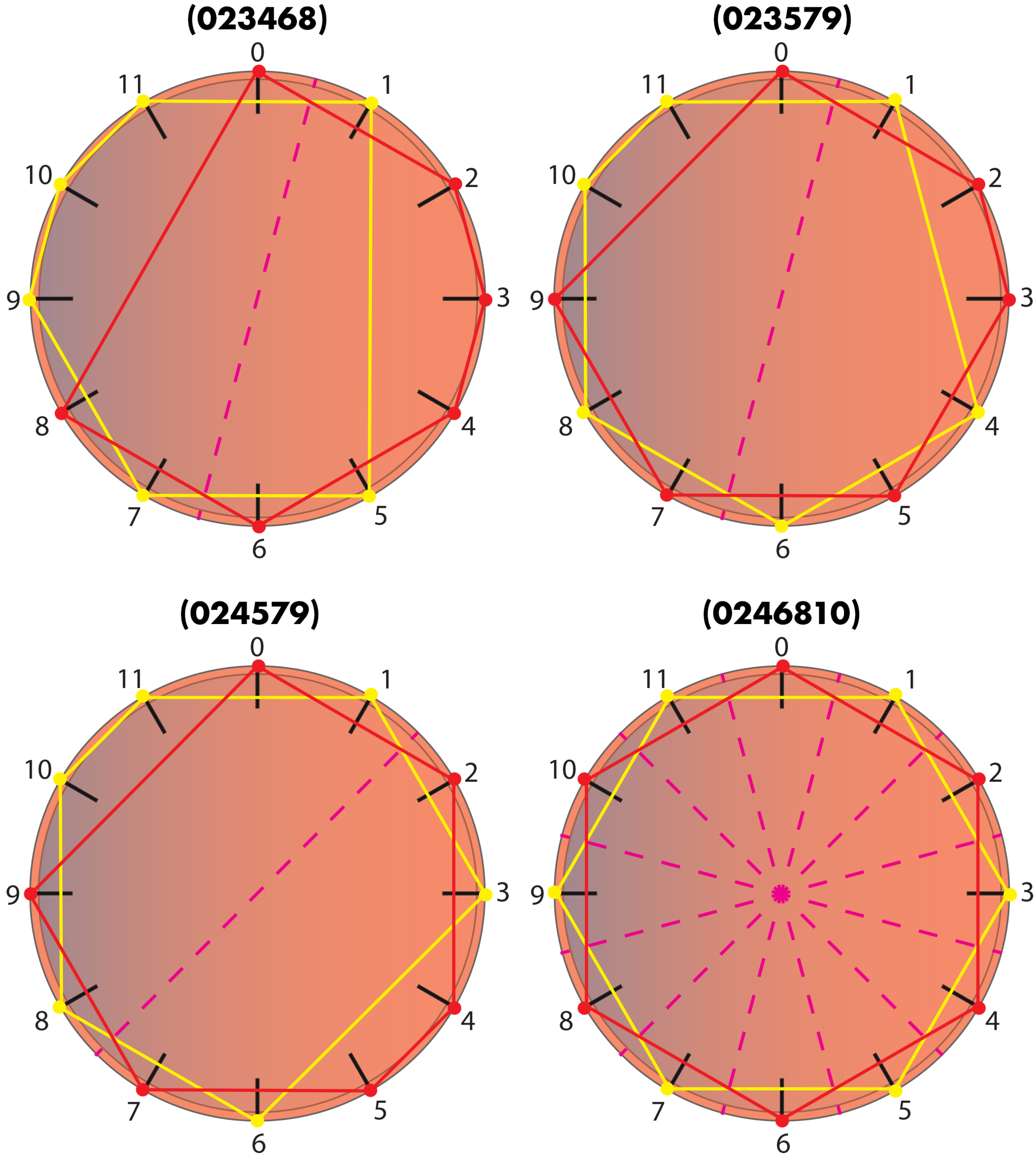
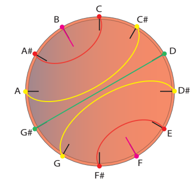


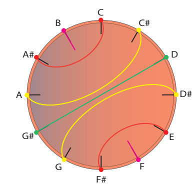
Figure 26. Twenty possible inversionally-symmetrical 6-onset rhythms on a12-pulse timeline.



CONCLUSION

This paper sought to reconcile the various and seemingly contradictory applications of the term *inversion* used in music. In the process, some new and fresh perspectives were offered that may well present new practical applications to the music student, improviser, and composer.

It is acknowledged that any theoretic approach to develop content in this manner is not music itself, but may perhaps facilitate the creation of more sophisticated and varied content within a language that is consistent, rich and expressive. If the author has aided a step towards this aspiration, this paper has humbly achieved its purpose. Thank you for reading and sharing.



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